Graph Theoretic Methods for the Computation of Disturbance Decoupling Feedback Matrices for Structured Systems

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ABSTRACT

Structured systems are considered for which the disturbance decoupling problem is known to be generically solvable. It is recalled how this generic solvability is in one-one correspondence with certain properties of the graph representing the structure of the system. Further, a graph oriented method is derived for the efficient computation of a feedback that actually solves the disturbance decoupling problem. It is believed that the method is especially useful in case of sparse system matrices. The method may also be advantageous in case some of the entries in the system matrices have an unknown value since the influence of the various entries is easy to establish.

1. INTRODUCTION

In this paper we study linear systems for which each entry in the system matrices either is fixed at zero or is a free parameter. We assume that the free parameters in a system are pairwise independent. Systems of the above type are often called structured systems and can be represented by means of directed graphs. In [16] we developed a graph theoretic characterization for the generic solvability of the disturbance decoupling problem formulated for structured systems. Here generic solvability is to be understood as solvability for almost all possible values for the free parameters. The conditions in [16] deal only with generic solvability, and nothing is said on what the associated feedback looks like.

In this paper we take the graph theoretic solvability conditions of [16] as a starting point and derive an algorithm for the computation of a feedback

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

solving the disturbance decoupling problem. In the algorithm we explicitly incorporate the graph that represents the zero-nonzero structure within the system matrices. The underlying idea is that the zero-nonzero structure of a system should be used as much as possible. The actual values of the entries in the system matrices should be used only in some final stage. Also, it may turn out that not all entries need to be known. Because of this our algorithm can be used in combination with a formula manipulation package in case the values of the free parameters are unknown or in case these parameters are seen as indeterminates. Finally, our algorithm can be advantageous in case the system matrices are sparse.

The algorithm developed here is in principle a nonrecursive version of the well-known structure algorithm (cf. [12]). However, the combination of the algorithm with graph theory is believed to be new. The algorithm is a generalization of the methods presented in [11] and is developed using ideas from [1].

The outline of the paper is as follows. In Section 2 we review some facts on the disturbance decoupling problem and we illustrate the main ideas behind the present paper. In Section 3 we present a version of the structure algorithm together with some modifications. We continue modifying the algorithm in Section 4. Therefore we introduce the notion of essential rows. We conclude Section 4 by stating a version of the algorithm that best suits our purposes and by relating it to the well-known input-output decoupling problem (cf. [5]). In Section 5 we describe how the zero-nonzero structure in the matrices of a structured system can be represented by means of a directed graph. Further we recall some important graph theoretic characterizations of for instance the generic rank of a transfer matrix and the generic solvability of the disturbance decoupling problem. Finally, we describe our main result: an algorithm for the efficient computation of a feedback solving the disturbance decoupling problem explicitly using the zero-nonzero structure in the system matrices. We illustrate the algorithm in Section 6 by means of two examples. In Section 7 we give some proofs, and in Section 8 we conclude the paper with some remarks.

2. DISTURBANCE DECOUPLING

In this paper we consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Qd(t), \\ z(t) &= Hx(t), \end{aligned} \tag{1}$$

with state $x(t) \in R^n$, control $u(t) \in R^m$, disturbance $d(t) \in R^q$, and output $z(t) \in R^p$. Here A, B, Q, and H are real matrices of suitable dimensions.

We recall that the disturbance decoupling problem for the system (1) amounts to finding a feedback $u(t) = Fx(t)$ such that the transfer matrix of the resulting closed loop system is zero, i.e. such that $H[sI - (A + BF)]^{-1}Q = 0$ for all s. For a fundamental treatment of the disturbance decoupling problem we refer to Wonham [14]. There it is shown that a solution to the problem exists if and only if $\operatorname{Im} Q \subseteq \mathscr{V}^*$, where $\mathscr{V}^*$ denotes the largest (A, B)-invariant subspace in $\operatorname{Ker} H$ (cf. [14]). We stress the fact that the inclusion $\operatorname{Im} Q \subseteq \mathscr{V}^*$ only is a condition on the *existence* of a feedback $u(t) = Fx(t)$ solving the disturbance decoupling problem. The matrix F itself does not appear in the condition. In [14] it is shown that any F such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$ will solve the problem. There it is also explained how first $\mathscr{V}^*$ can be computed, followed by the computation of F such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$. The computations are based on geometric techniques requiring full knowledge of A, B, and H.

In many cases many of the entries in the matrices are fixed at zero. Also, often some of the entries in A, B, and H have an unknown value. The purpose of this paper is to deal with these cases in an efficient way. Therefore, we derive an algorithm for the computation of a matrix F such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$ in which the zero-nonzero structure in the matrices A, B, and H is used, and in which, in first instance, the value of the free parameters (= nonzeros) is not relevant. In general, in the algorithm only a limited number of the free parameters will be actually needed. The value of the remaining free parameters is not important. The algorithm uses products of the form HA^iB and HA^{i+1}. The idea is that these products can be computed easily given the zero-nonzero structure of A, B, and H. Also, in computing these products the value of only a limited number of the free parameters may be required to be known.

We conclude this section by illustrating the ideas on which the paper is based. Therefore, we consider a system (1) with a very simple form:

$$A = \begin{bmatrix} 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \lambda_3 \\ 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 \\ 0 \\ \lambda_4 \end{bmatrix}, \qquad H = [\lambda_5 \quad 0 \quad 0].$$

It is easy to verify that for all λ_i, $i = 1, 2, 3, 4, 5$, with $\lambda_3 \neq 0$ the disturbance decoupling problem is solvable. Further, a feedback matrix achieving decoupling is given by $F = [0\ 0\ \ -\lambda_2/\lambda_3]$. Clearly, F is structured in the sense that it contains fixed zero entries. Moreover, F need only depend on two free parameters. Hence, the disturbance decoupling problem is generically solvable; a feedback matrix achieving decoupling may be structured and need only depend on a limited number of free parameters.

3. STRUCTURE ALGORITHM

In this section we present a nonrecursive version of the structure algorithm (cf. [12]). Throughout this paper we assume that rank $H(sI - A)^{-1}B = p$, i.e., $H(sI - A)^{-1}B$ has rank p for almost all s. In Section 7 we shall argue that in the context of the present paper this rank assumption can be made without loss of generality. We now denote

$$\Gamma = \begin{bmatrix} HB & 0 & \cdots & 0 \\ HAB & HB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ HA^{n-1}B & HA^{n-2}B & \cdots & HB \end{bmatrix} \in R^{pn \times mn},$$

$$\Lambda = \begin{bmatrix} HA \\ HA^2 \\ \vdots \\ HA^n \end{bmatrix} \in R^{pn \times n}.$$

The zeros in the Toeplitz matrix Γ denote zero matrices in $R^{p \times m}$. A matrix F such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$ can now be computed by means of the next algorithm.

ALGORITHM 3.1.

(a) Compute a matrix $K \in R^{p \times pn}$ such that $K\Gamma = [L, 0, \dots, 0] \in R^{p \times mn}$ with $L \in R^{p \times m}$ a full row rank matrix and $n - 1$ zero matrices in $R^{p \times m}$.

(b) Solve for $F \in R^{m \times n}$ in $M + LF = 0$ with $M = K\Lambda \in R^{p \times n}$.

Algorithm 3.1 will be proved in Section 7. The reason that we want to use Algorithm 3.1 is that it can be refined by incorporating additional information which can be easily obtained from the graph representing the zero-nonzero structure in the system matrices. To introduce these refinements we first somewhat rearrange the computations in Algorithm 3.1. For that purpose we denote by $\mathrm{col}(T_1, T_2, \ldots, T_r)$ the matrix obtained by stacking the matrices $T_1, T_2, \ldots, T_r$ on top of each other:

$$\mathrm{col}(T_1, T_2, \ldots, T_r) = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_r \end{bmatrix},$$

where the matrices $T_1, T_2, \ldots, T_r$ all have the same number of columns. For instance, we write $H = \mathrm{col}(h_1, h_2, \ldots, h_p)$ with rows $h_1, h_2, \ldots, h_p$. Furthermore, we denote for $i = 1, 2, \ldots, p$

$$\Gamma_i = \begin{bmatrix} h_i B & 0 & \cdots & 0 \\ h_i AB & h_i B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_i A^{n-1}B & h_i A^{n-2}B & \cdots & h_i B \end{bmatrix} \in R^{n \times mn},$$

$$\Lambda_i = \begin{bmatrix} h_i A \\ h_i A^2 \\ \vdots \\ h_i A^n \end{bmatrix} \in R^{n \times n}.$$

The matrix Γ_i is a Toeplitz matrix in which the zeros denote zero matrices in $R^{1 \times m}$. We note that the matrices Γ and $\mathrm{col}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)$ can be obtained from each other by elementary row permutations, i.e., there exists a permutation matrix P such that $P\Gamma = \mathrm{col}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)$. The permutation matrix P is also such that $P\Lambda = \mathrm{col}(\Lambda_1, \Lambda_2, \ldots, \Lambda_p)$. It is now immediate that Algorithm 3.1 can be modified as follows.

ALGORITHM 3.2.

(a) Compute matrices $K_1, K_2, \ldots, K_p \in R^{p \times n}$ such that $[K_1, K_2, \ldots, K_p]\,\mathrm{col}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p) = K_1\Gamma_1 + K_2\Gamma_2 + \cdots + K_p\Gamma_p = [L_1, 0, \ldots, 0] \in R^{p \times mn}$ with $L_1 \in R^{p \times m}$ a full row rank matrix and $n-1$ zero matrices in $R^{p \times m}$.

(b) Solve for $F \in R^{m \times n}$ in $M_1 + L_1 F = 0$ with $M_1 = [K_1, K_2, \ldots, K_p]\,\mathrm{col}(\Lambda_1, \Lambda_2, \ldots, \Lambda_p) = K_1\Lambda_1 + K_2\Lambda_2 + \cdots + K_p\Lambda_p$.

Below we shall use the fact that step (a) of the above algorithm can be written in more detail as follows:

(a) Compute matrices $K_1, K_2, \dots, K_p \in R^{p \times n}$ such that $[K_1,$ $_2, \dots, K_p]\mathrm{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p) = K_1\tilde{\Gamma}_1 + K_2\tilde{\Gamma}_2 + \cdots + K_p\tilde{\Gamma}_p = 0 \in R^{p \times m(n-1)}$ nd $[K_1, K_2, \dots, K_p]\mathrm{col}(\tilde{\Delta}_1, \tilde{\Delta}_2, \dots, \tilde{\Delta}_p) = K_1\tilde{\Delta}_1 + K_2\tilde{\Delta}_2 + \cdots + K_p\tilde{\Delta}_p = L_1 \in R^{p \times m}$ with L_1 a full row rank matrix.

Here we have denoted $\Gamma_i = [\tilde{\Delta}_i, \tilde{\Gamma}_i]$ for $i = 1, 2, \dots, p$, with

$$\tilde{\Delta}_i = \begin{bmatrix} h_i B \\ h_i AB \\ h_i A^2 B \\ \vdots \\ h_i A^{n-1} B \end{bmatrix} \in R^{n \times m},$$

$$\tilde{\Gamma}_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ h_i B & 0 & \cdots & 0 \\ h_i AB & h_i B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_i A^{n-2} B & h_i A^{n-3} B & \cdots & h_i B \end{bmatrix} \in R^{n \times (n-1)m}.$$

4. ESSENTIAL ROWS

As indicated before, our goal is to refine Algorithm 3.2 by incorporating information about the system that can be (easily) obtained from its graph. For that purpose, we need the notion of essential row (cf. [3]).

First we consider a general matrix M with rows $m_1, m_2, \dots, m_k$, i.e. $M = \mathrm{col}(m_1, m_2, \dots, m_k)$. We call the row m_i an *essential row* in M if the row m_i can *not* be written as a linear combination of the other rows $m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_k$ in M. A row of M that is not an essential row we call a *nonessential row*. For instance, any zero row is nonessential. Obviously, the row m_i is an essential row in M if and only if deleting m_i from M would give a drop in rank. Let $X \in R^{t \times k}$ be a solution of $XM = 0$, and denote $X = [x_1, x_2, \dots, x_k]$ with columns $x_1, x_2, \dots, x_k \in R^{t \times 1}$. Then, if m_i is an essential row in M, it is necessary that $x_i = 0 \in R^{t \times 1}$.

Next we consider in some detail the structure of the Toeplitz matrices Γ_i and $\tilde{\Gamma}_i$.

(a) First we note that the jth row of $\tilde{\Gamma}_i$ can be obtained from the jth row of Γ_i by simply deleting the leading (row) matrix $h_i A^{j-1}B$. Now consider the jth row of Γ_i as a row of $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$. It is clear that if the jth row of Γ_i is a nonessential row in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$, then the jth row of $\tilde{\Gamma}_i$ is a nonessential row in $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$. Conversely, if the jth row of $\tilde{\Gamma}_i$ is an essential row of $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$, so is the jth row of Γ_i in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$.

(b) Next we note that the jth row of Γ_i can be obtained from the $j+1$th row of $\tilde{\Gamma}_i$ by simply adding p zeros to its end. This means that if the $j+1$th row of $\tilde{\Gamma}_i$ is a nonessential row in $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$, then the jth row of Γ_i is a nonessential row in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$. Conversely, if the jth row of Γ_i is an essential row of $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$, so is the $j+1$th row of $\tilde{\Gamma}_i$ in $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$.

With the above observations we obtain the following.

PROPOSITION 4.1. *If the jth row of Γ_i is a nonessential row in* $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$, *then rows* 1 *up to j of Γ_i and $\tilde{\Gamma}_i$ are nonessential rows in* $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$ *and* $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$, *respectively, for i, j, $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant n$.*

Conversely, if the jth row of Γ_i is an essential row in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$, *then rows j up to n of Γ_i and rows $j+1$ up to n of $\tilde{\Gamma}_i$ are nonessential rows in* $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$ *and* $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$, *respectively, for i, j, $1 \leqslant i \leqslant p$ and $1 \leqslant j < n$.*

Recall that throughout this paper we assume that $H(sI - A)^{-1}B$ has full row rank. In [1] it has been shown that under this rank assumption the nth row of Γ_i is an essential row in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$ for all $i = 1, 2, \dots, p$. Therefore, it is possible to introduce the so-called essential orders defined as (cf. [1])

$$\beta_i = \min\left\{ j \,\middle|\, \text{row } j \text{ of } \Gamma_i \text{ is essential in } \operatorname{col}\left(\Gamma_1, \Gamma_2, \dots, \Gamma_p\right)\right\},$$
$$\text{for} \quad i = 1, 2, \dots, p.$$

By the above proposition rows β_i up to n of Γ_i and rows $\beta_i + 1$ up to n of $\tilde{\Gamma}_i$ are essential rows in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$ and $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$, respectively. Further, rows 1 up to $\beta_i - 1$ of both Γ_i and $\tilde{\Gamma}_i$ are nonessential rows in $\operatorname{col}(\Gamma_1, \Gamma_2, \dots, \Gamma_p)$ and $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_p)$, respectively.

Clearly, any essential row must be nonzero. To identify the nonzero rows in Γ_i and $\tilde{\Gamma}_i$ we introduce the so-called decoupling indices (cf. [5])

$$\alpha_i = \min\{j \mid h_i A^{j-1} B \neq 0\} \qquad \text{for} \quad i = 1, 2, \ldots, p.$$

By this definition rows 1 up to $\alpha_i - 1$ in Γ_i and rows 1 up to α_i in $\tilde{\Gamma}_i$ are zero rows. Also it is clear that $1 \leqslant \alpha_i \leqslant \beta_i \leqslant n$ for $i = 1, 2, \ldots, p$. Now we return to Algorithm 3.2.

The product $K_i \Gamma_i$ in step (a) of the algorithm can be computed by multiplying the jth column of K_i from the right with the jth row of Γ_i, and adding the products obtained. In the above we have seen that rows 1 up to $\alpha_i - 1$ in Γ_i are zero rows. Hence, for the product $K_i \Gamma_i$ columns 1 up to $\alpha_i - 1$ of K_i are not relevant and can be taken to be zero columns.

We claim that for the product $K_i \Gamma_i$ columns $\beta_i + 1$ up to n of K_i also are not relevant, since they are zero columns. To see this we recall that rows $\beta_i + 1$ up to n of $\tilde{\Gamma}_i$ are essential rows in $\operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_p)$. Because the matrices $K_1, K_2, \ldots, K_p$ are to be such that $[K_1, K_2, \ldots, K_p] \operatorname{col}(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_p) = 0$, it necessarily follows that columns $\beta_i + 1$ up to n of K_i are zero columns. With this our claim follows.

The above observations, combined with step (b) of Algorithm 3.2, make clear that without changing the outcome of the algorithm we can delete rows 1 up to $\alpha_i - 1$ and $\beta_i + 1$ up to n of Γ_i and Λ_i, for $i = 1, 2, \ldots, p$. In the matrix K_i we then have to delete columns 1 up to $\alpha_i - 1$ and $\beta_i + 1$ up to n, for $i = 1, 2, \ldots, p$. From this it follows that Algorithm 3.2 can be refined as follows.

ALGORITHM 4.2.

(a) Preliminary computation: For $i = 1, 2, \ldots, p$ compute the indices α_i, β_i and define

$$\hat{\Gamma}_i = \begin{bmatrix} h_i A^{\alpha_i - 1} B & 0 & \cdots & 0 & 0 & \cdots & 0 \\ h_i A^{\alpha_i} B & h_i A^{\alpha_i - 1} B & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ h_i A^{\beta_i - 1} B & h_i A^{\beta_i - 2} B & \cdots & h_i A^{\alpha_i - 1} B & 0 & \cdots & 0 \end{bmatrix}$$

$$\in R^{(\beta_i - \alpha_i + 1) \times \gamma m}$$

and

$$\hat{\Lambda}_i = \begin{bmatrix} h_i A^{\alpha_i} \\ h_i A^{\alpha_i + 1} \\ \vdots \\ h_i A^{\beta_i} \end{bmatrix} \in R^{(\beta_i - \alpha_i + 1) \times n},$$

where $\gamma = \max\{(\beta_i - \alpha_i + 1) \mid 1 \leqslant i \leqslant p\}$.

(b) Compute $\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_p \in R^{p \times (\beta_i - \alpha_i + 1)}$ such that $[\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_p] \operatorname{col}(\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots, \hat{\Gamma}_p) = \hat{K}_1 \hat{\Gamma}_1 + \hat{K}_2 \hat{\Gamma}_2 + \cdots + \hat{K}_p \hat{\Gamma}_p = [\hat{L}_1, 0, \ldots, 0] \in R^{p \times m\gamma}$ with $\hat{L}_1 \in R^{p \times m}$ a full row rank matrix and $\gamma - 1$ zero matrices in $R^{p \times m}$.

(c) Solve for $F \in R^{m \times n}$ in $\hat{M}_1 + \hat{L}_1 F = 0$ with $\hat{M}_1 = [\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_p] \operatorname{col}(\hat{\Lambda}_1, \hat{\Lambda}_2, \ldots, \hat{\Lambda}_p) = \hat{K}_1 \hat{\Lambda}_1 + \hat{K}_2 \hat{\Lambda}_2 + \cdots + \hat{K}_p \hat{\Lambda}_p$.

The index γ is introduced to avoid unnecessary zero columns in the matrix $\operatorname{col}(\hat{\Gamma}_1, \ldots, \hat{\Gamma}_p)$.

The reason for presenting Algorithm 4.2 as done above is that for almost all cases the values of the indices α_i and β_i can be extracted from the graph representing the zero-nonzero structure of the system. This graph, together with the combinatorial aspects of how to compute these indices, will be presented in the next section. After the indices α_i and β_i are determined, the matrices $\hat{\Gamma}_i$ and $\hat{\Lambda}_i$ have to be computed. The entries in these matrices are of the form $h_i A^k b_j$, where $B = [b_1, \ldots, b_m]$. In case the matrices A, B, and H are sparse, the graph representing the zero-nonzero structure of the system can be helpful in efficiently computing the above products. This will be discussed in some more detail in the next section.

Special Case

We conclude this section by considering the case that $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, p$. This means that every nonzero row of Γ_i is an essential row in $\operatorname{col}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)$. Step (a) of algorithm 4.2 yields that $\gamma = 1$, $\hat{\Gamma}_i = h_i A^{\alpha_i - 1} B$, $\hat{\Lambda}_i = h_i A^{\alpha_i}$ for $i = 1, 2, \ldots, p$. Step (b) then states that there are columns $\hat{K}_1, \ldots, \hat{K}_p \in R^p$ such that $[\hat{K}_1, \ldots, \hat{K}_p] \operatorname{col}(\hat{\Gamma}_1, \ldots, \hat{\Gamma}_p) = \hat{L}_1 \in R^{p \times m}$ with $\hat{L}_1$ a full row rank matrix. But this means that the matrix

$$\operatorname{col}\left(\hat{\Gamma}_1, \ldots, \hat{\Gamma}_p\right) = \begin{bmatrix} h_1 A^{\alpha_1 - 1} B \\ h_2 A^{\alpha_2 - 1} B \\ \vdots \\ h_p A^{\alpha_p - 1} B \end{bmatrix}$$

has full row rank and that $[\hat{K}_1, \ldots, \hat{K}_p]$ is an invertible matrix. From [5] it follows that the system is so-called input-output decoupable. This means that there exist matrices $F \in R^{m \times n}$ and $R \in R^{m \times p}$ such that $H[sI - (A + BF)]^{-1} BR$ is a nonsingular diagonal transfer matrix. It is easy to see from step

(c) in Algorithm 4.2 that, as in [5], the matrix F that suits our purposes can be computed from the equation

$$\begin{bmatrix} h_1 A^{\alpha_1 - 1} B \\ h_2 A^{\alpha_2 - 1} B \\ \vdots \\ h_p A^{\alpha_p - 1} B \end{bmatrix} F + \begin{bmatrix} h_1 A^{\alpha_1} \\ h_2 A^{\alpha_2} \\ \vdots \\ h_p A^{\alpha_p} \end{bmatrix} = 0.$$

5. GRAPHS

We recall that we here consider linear systems of the type (1) in which each of the entries in the system matrices either is fixed at zero or is an independent free parameter. As mentioned before, such systems can be represented by means of directed graphs. The directed graph which represents a structured system of the type (1) consists of a set of vertices V and a set of directed edges E and will be denoted by $G = (V, E)$.

The set V equals $X \cup U \cup D \cup Z$ with $X = \{x_1, \dots, x_n\}$ the set of state vertices, $U = \{u_1, \dots, u_m\}$ the set of input vertices, $D = \{d_1, \dots, d_q\}$ the set of disturbance vertices, and $Z = \{z_1, \dots, z_p\}$ the set of output vertices. Clearly, V consists of $n + m + q + p$ vertices.

Denoting by (v, v') a directed edge from the vertex $v \in V$ to the vertex $v' \in V$, the set of edges E is described by $E_A \cup E_B \cup E_Q \cup E_H$, where $E_A = \{(x_j, x_i) \mid a_{i,j} \neq 0\}$, $E_B = \{(u_j, x_i) \mid b_{i,j} \neq 0\}$, $E_Q = \{(d_j, x_i) \mid q_{i,j} \neq 0\}$, $E_H = \{(x_j, z_i) \mid h_{i,j} \neq 0\}$. Here, for instance, $a_{i,j} \neq 0$ means that the (i, j)th entry of A is a free parameter. For examples we refer to Section 6.

Now consider the graph $G = (V, E)$, and let W, W' be two disjoint nonempty subsets in V. We say that there exists a path from W to W' if there is an integer t and vertices $w_0, w_1, \dots, w_t \in V$ such that $w_0 \in W$, $w_t \in W'$, and $(w_{i-1}, w_i) \in E$ for all $i = 1, 2, \dots t$. The path is then said to have length t and to consist of the vertices $w_0, w_1, \dots, w_t$. Occasionally we denote the path by the sequence of directed edges it consists of, i.e. by (w_0, w_1), $(w_1, w_2), \dots, (w_{t-1}, w_t)$. Two paths from W to W' are called disjoint if the two paths consist of disjoint sets of vertices. We say that l paths from W to W' are pairwise disjoint if any two of them are disjoint. We say that the largest number of disjoint paths from W to W' is t if there exist t pairwise disjoint paths from W to W' and any set of $t + 1$ paths from W to W' contains at least two paths that are not disjoint.

Next, let W'' be an additional subset in V, and let a set of l pairwise disjoint paths from W to W' be given. If the intersection of the set W'' and the set of vertices that the l paths consist of has cardinality γ, then we say that the total number of vertices in W'' on the l paths is γ.

For a structured system of the type (1) and the associated graph $G = (V, E)$ we have the following results (cf. [10, 15, 16]).

THEOREM 5.1. *The generic rank of* $H(sI - A)^{-1}B$ *is equal to the largest number of pairwise disjoint paths from* U *to* Z.

For a structured system the generic values of the indices α_i and β_i can be characterized as follows (cf. [8, 1]). Here we denote $Z \setminus z_i = \{z \in Z \mid z \neq z_i\}$.

THEOREM 5.2.

(a) *The smallest total number of vertices contained in* X *on a path from* U *to* $\{z_i\}$ *is generically equal to* α_i, *for all* $i = 1, 2, \dots, p$.

(b) *Let* $H(sI - A)^{-1}B$ *generically have full row rank equal to* p. *The smallest total number of vertices contained in* X *on* p *pairwise disjoint paths from* U *to* Z minus *the smallest total number of vertices contained in* X *on* $p - 1$ *pairwise disjoint paths from* U *to* $Z \setminus z_i$ *is generically equal to* β_i, *for all* $i = 1, 2, \dots, p$.

The generic solvability of the disturbance decoupling problem for a structured system of the type (1) can be characterized as follows (cf. [16, 2])

THEOREM 5.3. *The disturbance decoupling problem for a structured system of the type* (1) *is generically solvable if and only if*

(a) *the largest number of pairwise disjoint paths from* U *to* Z *equals the largest number of pairwise disjoint paths from* $U \cup D$ *to* Z, *say* t, *and*

(b) *the smallest total number of vertices contained in* $X \cup U$ *on* t *pairwise disjoint paths from* U *to* Z *equals the smallest total number of vertices contained in* $X \cup U$ *on* t *pairwise disjoint paths from* $U \cup D$ *to* Z.

Since we focus here only on systems for which $H(sI - A)^{-1}B$ generically has full row rank, the next corollary easily follows.

COROLLARY 5.4. *The disturbance decoupling problem for a structured system of the type* (1) *is generically solvable if and only if the smallest total number of vertices contained in* $X \cup U$ *on* p *pairwise disjoint paths from* U *to* Z *equals the smallest total number of vertices contained in* $X \cup U$ *on* p *pairwise disjoint paths from* $U \cup D$ *to* Z.

By the above results we can establish the generic solvability of the disturbance decoupling problem for a structured system of the type (1) by performing some computations on its graph. We can do this in an efficient way by using algorithms from combinatorics based on the *max flow min cut* theorem and on the theory of *minimal cost flows* in transportation networks (cf. [6, Chapter 4; 13, Chapter 2]).

Next, if required we can determine from the system's graph the generic values of the indices α_i and β_i. Also here the abovementioned combinatoric algorithms can be used.

Following Algorithm 4.2, the next step is to compute the matrices $\hat{\Gamma}_i$ and $\hat{\Lambda}_i$ containing entries of the form $h_i A^k b_j$ and $h_i A^{k+1}$. In order to efficiently compute the products $h_i A^k b_j$ (similar remarks hold for the computation of $h_i A^{k+1}$) we denote by $P_{i,j}(k)$ the set of paths from u_j to z_i that have length $k + 2$. From the graph $G = (V, E)$ it is immediately clear that any path in $P_{i,j}(k)$ consists of the vertices u_j, z_i and at least one and at most $k + 1$ vertices in X. Now consider a path P in $P_{i,j}(k)$, and let it consist of the edges $(u_j, x_{\tau_0}), (x_{\tau_0}, x_{\tau_1}), \dots, (x_{\tau_{k-1}}, x_{\tau_k}), (x_{\tau_k}, z_i)$, where the indices $\tau_0, \tau_1, \dots, \tau_k$ are not necessarily distinct. Define $f(P) = b_{\tau_0, j} \prod_{t=1}^{k} a_{\tau_t, \tau_{t-1}} h_{i, \tau_k}$. In [8] it has been shown that $h_i A^k b_j = \sum_{P \in P_{i,j}(k)} f(P)$. Hence, the graph may also be used for the computation of the product $h_i A^k b_j$.

Combining all the above observations, we come to the next algorithm. Let a structured system of the type (1) be given, and let its graph be denoted by $G = (V, E)$.

ALGORITHM 5.5.

(a) Apply Theorem 5.3 to determine whether or not the disturbance decoupling problem for the system is generically solvable. If not, then we can stop; else we can proceed as described next. Assume that the largest number of pairwise disjoint paths from U to Z is p'. Select p' vertices from Z forming Z' such that the largest number of pairwise disjoint paths from U to Z' is p'. Renumber the vertices in Z in such a way that $Z' = \{z_1, \dots, z_{p'}\}$. Delete the vertices $z_{p'+1}, \dots, z_p$ and the edges incident with these vertices. For ease of notation drop the primes.

(b) Using the (reduced) graph G, now compute the generic values of the indices α_i and β_i, for all $i = 1, 2, \dots, p$, and compute, as indicated in Section 4, the matrices $\hat{\Gamma}_i$ and $\hat{\Lambda}_i$, for all $i = 1, 2, \dots, p$. For the efficient computation of these matrices the graph G may be used.

(c) Next perform step (b) and (c) of Algorithm 4.2 to obtain a feedback matrix F that is a solution to the disturbance decoupling problem. Note that only in steps (b) and (c) of Algorithm 4.2 are the values of (some of) the parameters required to be known.

The graph obtained after step (a) corresponds to a reduced structured system of the form $\dot{x}(t) = Ax(t) + Bu(t) + Qd(t)$, $z'(t) = H'x(t)$ with $H'(sI - A)^{-1}B$ generically of full row rank p'. From Section 7 it follows that the disturbance decoupling problem for the original system is generically solvable if and only if it is generically solvable for the reduced system. Furthermore, it is indicated there that for any F such that $H[sI - (A + BF)]^{-1}Q = 0$, also $H'[sI - (A + BF)]^{-1}Q = 0$, and conversely. Therefore, we may concentrate on the reduced system, and for ease of notation we drop the primes. Hence, we continue in step (b) with a structured system of the type (1) for which the disturbance decoupling problem is generically solvable and with $H(sI - A)^{-1}B$ generically of full rank p.

6. WORKED EXAMPLES

In this section we illustrate Algorithm 5.5 in some detail. We start by reconsidering the example in Section 2. Hence, $n = 3$, $m = q = p = 1$. The graph associated with the system can be depicted as in Figure 1. Note that $X = \{x_1, x_2, x_3\}$, $U = \{u_1\}$, $D = \{d_1\}$, and $Z = \{z_1\}$. Using the graph, it follows from Theorem 5.3 that the disturbance decoupling problem is generically solvable. Indeed, there is only one path from U to Z, and the two paths from $U \cup D$ to Z are not disjoint. Further, the only path from U to Z contains three vertices of $X \cup U$, while any two of the paths from $U \cup D$ to Z contain at least three vertices of $X \cup U$. By Theorem 5.2 it follows easily that $\alpha_1 = \beta_1 = 2$. As in the special case discussed at the end of Section 4, the feedback matrix F solving the disturbance decoupling problem can be computed by solving the equations $h_1 ABF + h_1 A^2 = 0$ where $h_1 = H$. Hence, F satisfies $\lambda_1\lambda_3\lambda_5 F + [0 \;\; 0 \;\; \lambda_1\lambda_2\lambda_5] = 0$. So $F = [0 \;\; 0 \;\; -\lambda_2/\lambda_3]$, as we also have seen in Section 2. Note that the above answer is obtained by

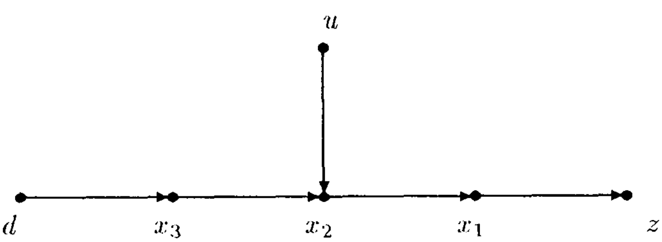


FIG. 1.

means of formula manipulation only and that no actual parameter values are required.

Next we consider a more complicated system of type (1) with system matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ \lambda_5 & 0 \\ 0 & 0 \\ \lambda_6 & \lambda_7 \\ 0 & 0 \\ 0 & \lambda_8 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} \lambda_9 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \lambda_{10} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & \lambda_{11} & \lambda_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{13} & 0 & 0 & \lambda_{14} \end{bmatrix}.$$

So $n = 7$, $m = q = p = 2$. The parameters $\lambda_1, \dots, \lambda_{14}$ are collected in a vector $\lambda \in R^{14}$. The structure of the system is given by the graph depicted in Figure 2. In the graph, the largest number of pairwise disjoint paths from U to Z and also from $U \cup D$ to Z is equal to 2. This implies that for almost all $\lambda \in R^{14}$ the transfer matrix $H(sI - A)^{-1}B$ has rank 2. Hence, $H(sI - A)^{-1}B$ generically has full rank. Also, it is easy to see that any pair of pairwise disjoint paths from U to Z and from $U \cup D$ to Z contains five or more vertices in $X \cup U$. From Theorem 5.3 it therefore follows that the disturbance decoupling problem is solvable for almost all $\lambda \in R^{14}$.

Next we have to compute the generic values of the decoupling indices α_1, α_2 and the essential orders β_1, β_2. Note that the shortest path in the graph from U to z_i has one vertex in X, for both $i = 1$ and $i = 2$. Furthermore, note that the smallest number of vertices in X in any pair of disjoint paths

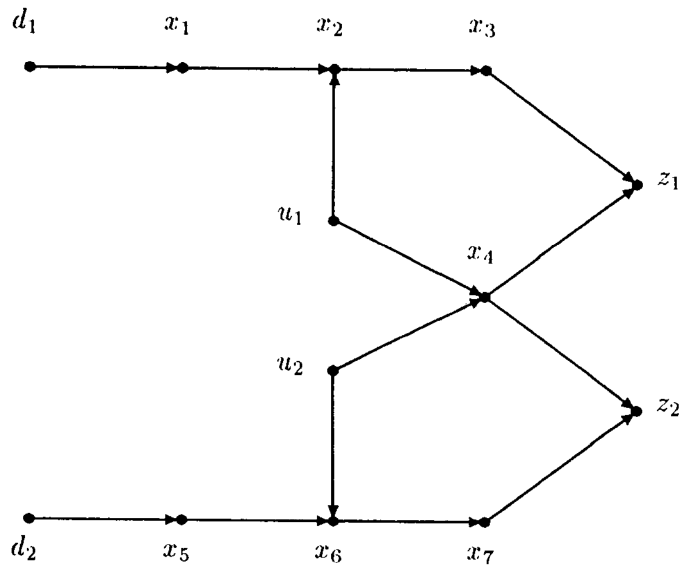


FIG. 2.

from U to Z is 3. From Theorem 5.2 it now follows that generically $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$.

Next the matrices $\hat{\Gamma}_1, \hat{\Gamma}_2$ and $\hat{\Lambda}_1, \hat{\Lambda}_2$ have to be computed. Note that $\gamma = 2$:

$$\hat{\Gamma}_1 = \begin{bmatrix} \lambda_6\lambda_{12} & \lambda_7\lambda_{12} & 0 & 0 \\ \lambda_2\lambda_5\lambda_{11} & 0 & \lambda_6\lambda_{12} & \lambda_7\lambda_{12} \end{bmatrix},$$

$$\hat{\Gamma}_2 = \begin{bmatrix} \lambda_6\lambda_{13} & \lambda_7\lambda_{13} & 0 & 0 \\ 0 & \lambda_4\lambda_8\lambda_{14} & \lambda_6\lambda_{13} & \lambda_7\lambda_{13} \end{bmatrix},$$

$$\hat{\Lambda}_1 = \begin{bmatrix} 0 & \lambda_2\lambda_{11} & 0 & 0 & 0 & 0 & 0 \\ \lambda_1\lambda_2\lambda_{11} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\Lambda}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_4\lambda_{14} & 0 \\ 0 & 0 & 0 & 0 & \lambda_3\lambda_4\lambda_{14} & 0 & 0 \end{bmatrix}.$$

The entries in the matrices easily can be determined from the graph. Following Algorithm 5.5, step (c), there remain steps (b) and (c) of Algorithm 4.2:

(b) Determine $\hat{K}_1$ and $\hat{K}_2$ such that $\hat{K}_1\hat{\Gamma}_1 + \hat{K}_2\hat{\Gamma}_2 = [\hat{L}_1, 0]$ with $\hat{L}_1 \in R^{2\times 2}$ full row rank.

(c) Solve for F in $\hat{M}_1 + \hat{L}_1 F = 0$ where $\hat{K}_1\hat{\Lambda}_1 + \hat{K}_2\hat{\Lambda}_2 = \hat{M}_1$.

Here we can for instance take

$$\hat{K}_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{\lambda_{13}}{\lambda_{12}} \end{bmatrix}, \qquad \hat{K}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then F such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$ and consequently solving the disturbance decoupling problem can be found from

$$\begin{bmatrix} \lambda_6\lambda_{13} & \lambda_7\lambda_{13} \\ -\dfrac{\lambda_2\lambda_5\lambda_{11}\lambda_{13}}{\lambda_{12}} & \lambda_4\lambda_8\lambda_{14} \end{bmatrix} F + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \lambda_4\lambda_{14} & 0 \\ -\dfrac{\lambda_1\lambda_2\lambda_{11}\lambda_{13}}{\lambda_{12}} & 0 & 0 & 0 & \lambda_3\lambda_4\lambda_{14} & 0 & 0 \end{bmatrix} = 0.$$

7. PROOFS

In this section we give mathematical justification of some earlier statements.

Rank Assumption

We denote $T(s) = H(sI - A)^{-1}B$ and $S(s) = H(sI - A)^{-1}Q$. We recall that throughout this paper we assume that the disturbance decoupling problem for the system (1) is solvable. This is equivalent to the existence of a strictly proper rational matrix $X(s)$ such that $T(s)X(s) = S(s)$ (cf. [7]). From this it follows that rank $T(s) = \operatorname{rank}[T(s) \;\; S(s)]$.

We claim that within the context of this paper the assumption that $T(s)$ has full row rank can be made without loss of generality. Indeed, suppose that $T(s)$ does not have full row rank, i.e. rank $T(s) = p' < p$. Then it is possible to find a permutation matrix P such that

$$PT(s) = \begin{bmatrix} T'(s) \\ T''(s) \end{bmatrix} \quad \text{and} \quad PS(s) = \begin{bmatrix} S'(s) \\ S''(s) \end{bmatrix}$$

with $T'(s)$ having full row rank p'. The matrix P can be simply determined by picking out p' rows of $T(s)$ that are linearly independent. The latter can be done by collecting p' outputs z_i into a vector z' such that the transfer matrix from u to z' has full row rank. This transfer matrix can be written as $H'(sI - A)^{-1}B$ with H' a $p' \times n$ matrix composed of p' suitably chosen rows from H. Clearly, we then may take $T'(s) = H'(sI - A)^{-1}B$.

Obviously, since $T'(s)$ has full row rank, it follows that $p' = \operatorname{rank} T'(s) = \operatorname{rank}[T'(s) \;\; S'(s)]$. Further, since $\operatorname{rank}[T'(s) \;\; S'(s)] = p' = \operatorname{rank} T(s) = \operatorname{rank}[T(s) \;\; S(s)] = \operatorname{rank} P[T(s) \;\; S(s)]$, the rows in $[T''(s) \;\; S''(s)]$ are linearly dependent on the rows in $[T'(s) \;\; S'(s)]$. This means that there is a rational matrix $Y(s)$ such that $Y(s)T'(s) = T''(s)$ and $Y(s)S'(s) = S''(s)$.

We therefore obtain that any solution $X(s)$ to the equation $T(s)X(s) = S(s)$ is also a solution to the equation $T'(s)X(s) = S'(s)$, and conversely. It follows now from [7] that the disturbance decoupling problem is solvable for $\dot{x}(t) = Ax(t) + Bu(t) + Qd(t)$, $z(t) = Hx(t)$ if and only if it is solvable for $\dot{x}(t) = Ax(t) + Bu(t) + Qd(t)$, $z'(t) = H'x(t)$. Using the identity $H[sI - (A + BF)]^{-1}Q = T(s)[I - F(sI - A)^{-1}B]^{-1}F(sI - A)^{-1}Q + S(s)$ and a similar identity involving H' and $T'(s)$, $S'(s)$, it easily follows from the above that if F is such that $H[sI - (A + BF)]^{-1}Q = 0$, then $H'[sI - (A + BF)]^{-1}Q = 0$, and conversely. Hence, for the computation of a feedback matrix F solving the disturbance decoupling problem we may restrict ourselves to the part for which our rank assumption holds.

Factorization

Under our rank assumption the matrix $T(s)$ can be factorized as $T(s) = U(s)\Sigma(s)L(s)$ (cf. [4]). In this factorization $U(s)$ is a $p \times p$ proper rational matrix that has a proper rational inverse, $L(s)$ is a $p \times m$ proper rational matrix that has a proper rational right inverse, and $\Sigma(s) = \operatorname{diag}(s^{-n_1}, s^{-n_2}, \dots, s^{-n_p})$ with $0 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_p$. The integers $n_1, n_2, \dots, n_p$ are unique and are called the infinite zero orders of $T(s)$. From the behavior at infinity it easily follows that $\sum_{i=1}^{p} n_i \leqslant n$.

Proof of Algorithm 3.1. Next we give a proof of Algorithm 3.1 starting from the above factorization. From the factorization it follows that $K(s) := [U(s)\Sigma(s)]^{-1}$ exists and has a Laurent series expansion of the form $K(s) = K_{-n}s^n + K_{-(n-1)}s^{n-1} + \cdots + K_{-1}s^1 + K_0 + K_1 s^{-1} + K_2 s^{-2} + \cdots$. Let the Laurent series expansions of $L(s)$ and $T(s)$ be given by $L(s) = L_0 + L_1 s^{-1} + L_2 s^{-2} + \cdots$ and $T(s) = HBs^{-1} + HABs^{-2} + HA^2Bs^{-3} + \cdots$, respectively. Then $L_0 = L(\infty)$ has full row rank equal to p. Now consider the identity $L(s) = K(s)T(s)$ for the individual nonnegative powers of s. This yields n equations which can be written as follows.

$$[K_{-1}, K_{-2}, \dots, K_{-n}]\begin{bmatrix} HB & 0 & \cdots & 0 \\ HAB & HB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ HA^{n-1}B & HA^{n-2}B & \cdots & HB \end{bmatrix}$$

$$= [L_0, 0, \dots, 0, 0]. \tag{2}$$

Next define $M(s) = K(s)H(sI - A)^{-1}$. The Laurent series expansion of $H(sI - A)^{-1}$ is given by $H(sI - A)^{-1} = Hs^{-1} + HAs^{-2} + HA^2s^{-3} + \cdots$. From this and the previous expansion of $K(s)$ it follows that the Laurent series expansion of $M(s)$ is of the form $M(s) = M_{-(n-1)}s^{n-1} + M_{-(n-2)}s^{n-2} + \cdots + M_{-1}s^1 + M_0 + M_1s^{-1} + M_2s^{-2} + \cdots$ with, for instance, $M_{-(n-1)} = K_{-n}H$, $M_{-(n-2)} = K_{-(n-1)}H + K_{-n}HA$, and $M_0 = K_{-n}HA^{n-1} + K_{-(n-1)}HA^{n-2} + \cdots + K_{-2}HA + K_{-1}H$.

Note that $M(s)B = K(s)T(s) = L(s)$ is proper. Hence, $M_{-(n-1)}B = M_{-(n-2)}B = \cdots = M_{-1}B = 0$ and $M_0B = L_0$. Furthermore, since $H(sI - A)^{-1} = Hs^{-1} + H(sI - A)^{-1}As^{-1}$, it follows that $M(s)\bar{x} = M(s)s^{-1}A\bar{x}$ for any $\bar{x} \in \operatorname{Ker} H$. The above expansion of $M(s)$ therefore implies that $M_{-(i-1)}\bar{x} = M_{-i}A\bar{x}$ for all $\bar{x} \in \operatorname{Ker} H$ and $i \leqslant n - 1$.

Now recall that $\mathscr{V}^*$ equals the set of vectors $\bar{x} \in \operatorname{Ker} H$ for which there exists a strictly proper rational vector $\omega(s)$ such that $H(sI - A)^{-1}\bar{x} + H(sI - A)^{-1}B\omega(s) = 0$ (cf. [7]). Obviously, it follows that $\mathscr{V}^*$ equals the set of $\bar{x} \in \operatorname{Ker} H$ for which there exists a strictly proper rational vector $\omega(s)$ such that $M(s)\bar{x} + L(s)\omega(s) = 0$. We claim that $\mathscr{V}^* = \bigcap_{i=0}^{n-1} \operatorname{Ker} M_{-i}$.

To prove this claim, let $\bar{x} \in \mathscr{V}^*$, and let $\omega(s)$ be a strictly proper rational vector such that $M(s)\bar{x} + L(s)\omega(s) = 0$. Since $L(s)\omega(s)$ is strictly proper, it follows necessarily that $M_{-i}\bar{x} = 0$ for $i = 0, 1, \dots, n - 1$. Hence, $\bar{x} \in \bigcap_{i=0}^{n-1} \operatorname{Ker} M_{-i}$. Conversely, let $\bar{x} \in \bigcap_{i=0}^{n-1} \operatorname{Ker} M_{-i}$. Because L_0 has full row rank, the strictly proper rational vector $\omega(s) = \sum_{i \geqslant 1} \omega_i s^{-i}$ such that $M(s)\bar{x} + L(s)\omega(s) = 0$ can be defined recursively by $L_0\omega_i = M_i\bar{x} - (L_1\omega_{i-1} + L_2\omega_{i-2} + \cdots + L_{i-1}\omega_1)$. This implies that $\bar{x} \in \mathscr{V}^*$, and our claim is proved.

Recall that $\mathscr{V}^*$ is an (A, B)-invariant subspace. Hence, for any $\bar{x} \in \mathscr{V}^*$ there are $\bar{y} \in \mathscr{V}^*$ and $\bar{u} \in R^m$ such that $A\bar{x} = \bar{y} + B\bar{u}$. Given such $\bar{x}$, $\bar{y}$, and $\bar{u}$, it follows from the above that $M_{-i}A\bar{x} = M_{-i}\bar{y} + M_{-i}B\bar{u} = 0$ for $i = 1, 2, \dots, n - 1$ and $M_0A\bar{x} = M_0\bar{y} + M_0B\bar{u} = L_0\bar{u}$.

Our goal is to compute a matrix F such that $(A + BF)\mathscr{V}^* \subseteq \mathscr{V}^*$. Since $\mathscr{V}^* = \bigcap_{i=0}^{n-1} \operatorname{Ker} M_{-i}$, such a matrix F has to be such that $M_{-i}(A +$

$BF)\bar{x} = 0$ for all $\bar{x} \in \mathscr{V}^*$ and $i = 0, 1, \dots, n-1$. Now observe that for any $\bar{x} \in \mathscr{V}^* \subseteq \operatorname{Ker} H$ there should hold

$$\begin{bmatrix} M_{-(n-1)} \\ \vdots \\ M_{-1} \\ M_0 \end{bmatrix} (A + BF)\bar{x} = \begin{bmatrix} M_{-(n-1)}A \\ \vdots \\ M_{-1}A \\ M_0 A \end{bmatrix} \bar{x} + \begin{bmatrix} M_{-(n-1)}B \\ \vdots \\ M_{-1}B \\ M_0 B \end{bmatrix} F\bar{x}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Since $M_{-i}B = 0$ and $M_{-i}A\bar{x} = 0$ for all $\bar{x} \in \mathscr{V}^*$ and for all $i = 1, 2, \dots, n-1$, the matrix F has to be such that $(M_0 A + M_0 BF)\bar{x} = 0$ for all $\bar{x} \in \mathscr{V}^*$. Because $M_0 B = L_0$ has full row rank, such F can be taken to be the solution of $M_0 A + L_0 F = 0$. Recall that $M_0 A = [K_{-1}, K_{-2}, \dots, K_{-n}] \operatorname{col}(HA, HA^2, \dots, HA^{n-1}, HA^n)$, where $K_{-1}, K_{-2}, \dots, K_{-n}$ satisfy (2) with L_0 full row rank. This completes the proof of Algorithm 3.1. ∎

8. REMARKS AND CONCLUSIONS

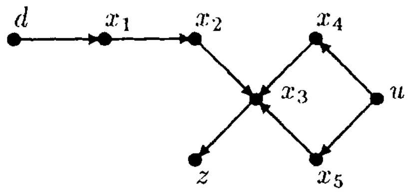
In this paper we have studied the disturbance decoupling problem. We have developed an algorithm (Algorithm 5.5) for computing a feedback matrix that solves the problem, in which we explicitly make use of the zero-nonzero structure present in the system matrices. The algorithm is based on a representation of the zero-nonzero structure by means of a directed graph, and it makes use of well-known and efficient algorithms from combinatorics. Clearly, the algorithm may be useful in case the nonzero entries in the system matrices are not all exactly known.

We note that generically Algorithm 5.5 produces the right answer. This means that for almost all possible values of the nonzero entries in the system marices the algorithm comes up with the correct feedback. This also implies however that there may be a set of zero Lebesgue measure of nonzero entries for which the result of the algorithm is not correct.

In the first two steps of Algorithm 5.5, the zero-nonzero structure in the system matrices is used and wrong conclusions may be drawn because generic properties do not coincide with the properties of the particular (numerically specified) system under consideration. Below we give examples of such situations. We assume therefore that the nonzero entries in the system matrices are each parametrized by a real parameter λ_i, $i = 1, 2, \dots, N$, with N the number of nonzero entries. Further, we write $\lambda = \mathrm{col}(\lambda_1, \lambda_2, \dots, \lambda_N) \in R^N$ for the vector containing all the parameters. Then the class of systems with system matrices all having the same zero-nonzero structure as the system under consideration can be parametrized by λ. Generic properties are then properties that hold for almost all $\lambda \in R^N$, whereas properties of a particular system are properties that hold for a specific $\lambda \in R^N$.

(a): To indicate why in exceptional cases in step (a) of the algorithm the wrong conclusion can be drawn, we show below that the generic solvability of the disturbance decoupling problem and its solvability for a particular system do not imply each other (cf. [9, Section 4]).

We start by showing that generic solvability does not always imply solvability for a particular system, and we consider a system with a zero-nonzero structure depicted as follows:



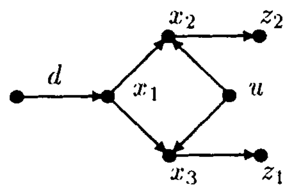
The system has the following system matrices ($N = 8$):

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \lambda_5 \\ \lambda_6 \end{bmatrix},$$

$$Q = \begin{bmatrix} \lambda_7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad H = [0 \quad 0 \quad \lambda_8 \quad 0 \quad 0].$$

As done for the very first example of this paper, we can verify by means of Theorem 5.3 that for this example the disturbance decoupling problem is generically solvable, i.e. is solvable for almost all $\lambda \in R^8$. However, there is a set of zero Lebesgue measure of parameter vectors $\lambda \in R^8$ for which the disturbance decoupling problem is not solvable. We can obtain a vector in this set by taking λ_i such that $\lambda_3\lambda_5 + \lambda_4\lambda_6 = 0$. Further, we may take $\lambda_i \neq 0$ for $i = 1, 2, \ldots, 8$. Then it follows immediately that $HA^iB = 0$ for all $i \geqslant 0$ and $HA^2Q \neq 0$. This means that $T(s) = 0$ while $S(s) \neq 0$. Hence, the disturbance decoupling problem is not solvable for the system with this particular $\lambda \in R^8$, although the problem is generically solvable for the class of systems that have the above zero-nonzero structure.

Next we show that the generic unsolvability of the disturbance decoupling problem does not always imply the unsolvability of the problem for a particular system. To show this we start from a system with zero-nonzero structure as depicted in the graph below:



The system matrices are as follows ($N = 7$):

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_1 & 0 & 0 \\ \lambda_2 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \lambda_3 \\ \lambda_4 \end{bmatrix},$$

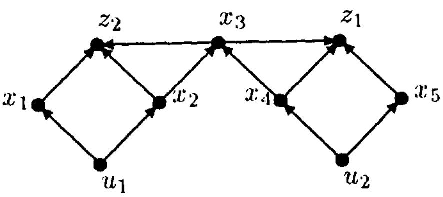
$$Q = \begin{bmatrix} \lambda_5 \\ 0 \\ 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & \lambda_6 & 0 \\ 0 & 0 & \lambda_7 \end{bmatrix}.$$

Using Theorem 5.3, it follows that the disturbance decoupling problem for this system is generically unsolvable. For instance, this results from the fact that generically rank $T(s) = 1$ while rank $[T(s) \;\; S(s)] = 2$ (see Theorem 5.1). Theorem 5.3 now implies that the disturbance decoupling problem is unsolvable (not solvable) for almost all parameter vectors $\lambda \in R^7$. However, there is a set of zero Lebesgue measure of parameter vectors $\lambda \in R^7$ for which the disturbance decoupling problem is solvable. We can obtain a vector in this set by taking λ_i such that $\lambda_1\lambda_4 - \lambda_2\lambda_3 = 0$. Further, we may take $\lambda_i \neq 0$, $i = 1, 2, \ldots, 7$. Then, by applying the feedback

$$F = \begin{pmatrix} -\lambda_1/\lambda_3 & 0 & 0 \\ -\lambda_2/\lambda_4 & 0 & 0 \end{pmatrix},$$

it follows that both $A + BF$ and HQ are zero matrices and consequently that $H[sI - (A + BF)]^{-1}Q = HQ\, s^{-1} = 0$. Hence, the disturbance decoupling problem for the system with this particular $\lambda \in R^7$ is solvable although the problem is generically unsolvable for the class of systems with the above zero-nonzero structure.

(b): We now show how in step (b) of Algorithm 5.5 the wrong conclusions may be drawn because the generic values of the decoupling indices and the essential orders do not correspond with their values for a particular system. To show this we consider a system of which the zero-nonzero structure can be depicted as follows:



The system matrices are as follows ($N = 12$):

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda_3 & 0 \\ \lambda_4 & 0 \\ 0 & 0 \\ 0 & \lambda_5 \\ 0 & \lambda_6 \end{bmatrix},$$

$$H = \begin{bmatrix} \lambda_7 & \lambda_8 & \lambda_9 & 0 & 0 \\ 0 & 0 & \lambda_{10} & \lambda_{11} & \lambda_{12} \end{bmatrix}.$$

It is immediate that

$$HB = \begin{pmatrix} \lambda_3\lambda_7 + \lambda_4\lambda_8 & 0 \\ 0 & \lambda_5\lambda_{11} + \lambda_6\lambda_{12} \end{pmatrix}, \qquad HAB = \begin{pmatrix} \lambda_1\lambda_4\lambda_9 & \lambda_2\lambda_5\lambda_9 \\ \lambda_1\lambda_4\lambda_{10} & \lambda_2\lambda_5\lambda_{10} \end{pmatrix},$$

and HA^iB is a 2×2 zero matrix for all $i \geqslant 2$. It follows from Theorem 5.1 that generically the rank of $T(s)$ equals 2. Theorem 5.2 yields that generically $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$. Here we have written α_i and β_i, $i = 1, \dots, p$

with $p = 2$, for the generic values of the decoupling indices and the essential orders, respectively. There is a set of zero Lebesgue measure of parameter vectors $\lambda \in R^{12}$ for which some of the above indices have different (larger) values. To obtain a vector in this set we can take λ_i such that $\lambda_3\lambda_7 + \lambda_4\lambda_8 = 0$. Further, we may take $\lambda_i \neq 0$, $i = 1, 2, \ldots, 12$, and $\lambda_5\lambda_{11} + \lambda_6\lambda_{12} \neq 0$. Then it follows from the definitions in Section 4 that for this particular λ the above indices have the values $\bar{\alpha}_1 = 2$, $\bar{\alpha}_2 = 1$ and $\bar{\beta}_1 = 2$, $\bar{\beta}_2 = 1$. Here $\bar{\alpha}_i$ and $\bar{\beta}_i$, $i = 1, \ldots, p$ (with $p = 2$), denote the values of the decoupling indices and the essential orders, respectively, for a particular λ.

Combining the definition of the decoupling indices, Theorem 5.2(a), and the remark about the use of the graph in the computation of the product $h_i A^k b_j$ just before Algorithm 5.5, it easily follows that always $\alpha_i \leqslant \bar{\alpha}_i$ for all $i = 1, 2, \ldots, p$. Hence, the generic value of a decoupling index is never larger than its value for a λ corresponding to a particular system. As follows from the above example, we have a strict inequality when cancellation occurs.

For essential orders, similar statements concerning their generic value and their value for a particular system are more difficult to give. These statements are omitted here to avoid technicalities and also because the above example already indicates in which cases Algorithm 5.5 may give poor results.

Algorithm 5.5 may not work properly when based on generic arguments too many rows are deleted from the matrix $\mathrm{col}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)$. This happens if $\bar{\alpha}_i < \alpha_i$ for some $i = 1, 2, \ldots, p$ or $\beta_i < \bar{\beta}_i$ for some $i = 1, 2, \ldots, p$ or both.

As we have seen, the first can never occur. Moreover, we have seen in the above example that there exist systems such that $\alpha_i < \bar{\alpha}_i$ for some $i = 1, 2, \ldots, p$, implying that on the basis of generic arguments less rows than allowed are deleted. The possibly nondeleted rows are zero rows and do not affect the outcome of step (c) of Algorithm 5.5. We have also seen in the above example that it may happen that $\beta_i < \bar{\beta}_i$ for some $i = 1, 2, \ldots, p$. This means that on the basis of generic arguments more rows than allowed are deleted. The deletion of these extra rows from $\mathrm{col}(\Gamma_1, \Gamma_2, \ldots, \Gamma_p)$ can make Algorithm 5.5 come up with a wrong feedback in step (c), if the feedback can be computed at all.

Hence, Algorithm 5.5 can go wrong if there is an essential order whose generic value is less than its value for the particular system under consideration. We do want to stress again that given the zero-nonzero structure of the system, such situations are exceptional (nongeneric), and that generically Algorithm 5.5 will work properly.

The author would like to thank the referee for raising questions leading to the major part of the last section.

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Received 5 August 1992; final manuscript accepted 21 December 1992